

POSTULATION OF DISJOINT UNIONS OF LINES AND A MULTIPLE POINT, II

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ABSTRACT. We study the postulation of a general union $X \subset \mathbb{P}^3$ of one m-point mP and t disjoint lines. We prove that it has the expected Hilbert function, proving a conjecture by E. Carlini, M. V. Catalisano and A. V. Geramita.

1. INTRODUCTION

A scheme $X \subset \mathbb{P}^r$ is said to have *maximal rank* if for all integers $t > 0$ the restriction map $H^0(\mathcal{O}_{\mathbb{P}^r}(t)) \rightarrow H^0(X, \mathcal{O}_X(t))$ is either injective or surjective, i.e. if either $h^0(\mathcal{I}_X(t)) = 0$ or $h^1(\mathcal{I}_X(t)) = 0$, i.e. if X imposes the “expected” number of conditions to the vector space of all homogeneous degree t polynomials in $r + 1$ variables. R. Hartshorne and A. Hirschowitz proved that for all integers $t > 0$ and $r \geq 3$ a general union $X \subset \mathbb{P}^r$ of t general lines has maximal rank. E. Carlini, M. V. Catalisano and A. V. Geramita considered several cases in which we allow unions of linear spaces with certain multiplicities [2], [3], [4]. We recall that for each $P \in \mathbb{P}^r$ the m-point mP of \mathbb{P}^r is the closed subscheme of \mathbb{P}^r with $(\mathcal{I}_P)^m$ as its ideal sheaf. E. Carlini, M. V. Catalisano and A. V. Geramita proved that for all $r \geq 4$, $m > 0$ and $d > 0$ a general union of an m-point and d disjoint lines has maximal rank ([4]). In the case $r = 3$ they proved that there are some exceptional cases (the one with $2 \leq d \leq m$ and $t = m$); in [4] the failure of maximal rank for these cases is exactly described, i.e. all positive integers $h^0(\mathcal{I}_X(t))$ and $h^1(\mathcal{I}_X(t))$ are computed ([4, Theorem 4.2, part (ii)]). They conjectured in [4] that these are the only exceptional cases and proved the conjecture in some cases (e.g. if $m = 2$ by [4, Theorem 4.2, part (i)(e)]). In [1] their conjecture was proved when $m = 3$ and an asymptotic result was proved for arbitrary m ([1, Propositions 1 and 2]). In this paper we prove their conjecture in the case $m = 3$, i.e. we prove the following result.

Theorem 1. *Fix integers $m \geq 2$, $t > 0$ and $d > 0$. If $2 \leq d \leq m$, then assume $t \geq m + 1$. Let $Y \subset \mathbb{P}^3$ be a general union of d lines. Then either $h^1(\mathcal{I}_{mP \cup Y}(t)) = 0$ or $h^0(\mathcal{I}_{mP \cup Y}(t)) = 0$.*

A crucial step of the proof is contained in [4, Theorem 4.2, part (i)(c)]: the proof of the case $d = m + 2$ and $t = m + 1$. Let $Y \subset \mathbb{P}^3$ be a general union of $m + 2$ lines. They proved that $h^i(\mathcal{I}_{mP \cup Y}(m + 1)) = 0$, $i = 0, 1$. After [7] and [6] it is well-known that if certain crucial curves or unions of curves and points, say X_1 and

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X_2 , have $h^i(\mathcal{I}_{X_1}(t_0)) = 0$, $i = 0, 1$, and $h^i(\mathcal{I}_{X_2}(t_0 + 1)) = 0$, $i = 0, 1$, then it should be easy to control the postulation of all curves of degree $\geq \deg(X_2)$ with respect to all forms of degree $\geq t_0 + 2$. In our case by [4, Theorem 4.2, part (i)(c)] we may take $X_1 = mP \cup Y$ with $\deg(Y) = m + 2$. The key part of the proof is the construction of a good X_2 for $t_0 + 1 = m + 2$ and then to control the cases $t = t_0 + 3$ and $t = t_0 + 4$.

We work over an algebraically closed field \mathbb{K} . As far as we understand none of our quotations of [4] require the characteristic zero assumption made in [4].

2. PRELIMINARIES

For any integer $d > 0$ let $L(d)$ be the set of all unions $Y \subset \mathbb{P}^3$ of d disjoint lines. For any $P \in \mathbb{P}^3$ set $L(P, d) := \{Y \in L(d) : P \notin Y\}$. If P is a smooth point of a scheme T let $\{mP, T\}$ be the closed subscheme of T with $(\mathcal{I}_{P,T})^m$ as its ideal sheaf. We write mP instead of $\{mP, \mathbb{P}^3\}$. For any positive-dimensional $A \subseteq \mathbb{P}^3$ and any smooth point O of A a *tangent vector* of A with O as its support is a degree 2 connected zero-dimensional scheme $v \subset A$ such that $\deg(v) = 2$ and $v_{\text{red}} = \{O\}$.

Let $F \subset \mathbb{P}^3$ be any surface. Set $t := \deg(F)$. For each closed subscheme $Z \subset \mathbb{P}^3$ let $\text{Res}_F(Z)$ denote the residual scheme of Z with respect to F , i.e. the closed subscheme of \mathbb{P}^3 with $\mathcal{I}_Z : \mathcal{I}_F$ as its ideal sheaf. If Z is reduced, then $\text{Res}_F(Z)$ is the union of the irreducible components of Z not contained in F . Now assume $Z = mP$ for some $m > 0$ and some $P \in \mathbb{P}^3$. If $P \notin F$, then $\text{Res}_F(mP) = mP$. If P is a smooth point of F , then $\text{Res}_F(mP) = (m - 1)P$ (with the convention $0P = \emptyset$). For any integer $x \geq t$ we have an exact sequence

$$0 \rightarrow \mathcal{I}_{\text{Res}_F(Z)}(x - t) \rightarrow \mathcal{I}_Z(x) \rightarrow \mathcal{I}_{Z \cap F}(x) \rightarrow 0$$

Hence

$$\begin{aligned} \bullet \quad h^0(\mathcal{I}_Z(x)) &\leq h^0(\mathcal{I}_{\text{Res}_F(Z)}(x - t)) + h^0(F, \mathcal{I}_{Z \cap F}(x)); \\ \bullet \quad h^1(\mathcal{I}_Z(x)) &\leq h^1(\mathcal{I}_{\text{Res}_F(Z)}(x - t)) + h^1(F, \mathcal{I}_{Z \cap F}(x)). \end{aligned}$$

As in [2], [3, Lemma 3.3] and [4] we will call “the Castelnuovo’s inequality” any of these two inequalities. If F is either a plane or a smooth quadric, D is an effective divisor of F and $Z \subset F$ is a closed subscheme of F , $\text{Res}_D(Z)$ is the closed subscheme of F with $\mathcal{I}_{Z,F} : \mathcal{I}_{D,F}$ as its ideal sheaf (of course, $\mathcal{I}_{D,F} \cong \mathcal{O}_F(-D)$ as abstract line bundles on F). We also have the corresponding Castelnuovo’s exact sequence of Res_D and the associated Castelnuovo’s inequalities.

Set $0P := \emptyset$. We use the convention that $\binom{t}{3} = 0$ if $-2 \leq t \leq 2$ and $\binom{t+2}{2} = 0$ if $-1 \leq t \leq 1$. We have $\deg(0P) = 0 = \binom{2}{3}$. For all integers $m \geq 0$ and $k \geq 0$ define the integers $a_{m,k}$ and $b_{m,k}$ by the relations

$$(1) \quad \binom{m+1}{3} + (k+1)a_{m,k} + b_{m,k} = \binom{k+3}{3}, 0 \leq b_{m,k} \leq k$$

If $k \geq 2$ from (1) for $k, k-2, k-1$ and $m-1$ we get

$$(2) \quad 2a_{m,k-2} + (k+1)(a_{m,k} - a_{m,k-2}) + b_{m,k} - b_{m,k+1} = (k+1)^2$$

Taking the difference of (2) with $k = m+2$ and the same equation with $(m', k') = (m-1, m+1)$ and using that $\binom{m+2}{3} - \binom{m+1}{3} = \binom{m+1}{2}$ and $\binom{m+4}{2} - \binom{m+1}{2} = 3m+6$, we get

$$(3) \quad a_{m-1,m+1} + (m+3)(a_{m,m+2} - a_{m-1,m+2}) + b_{m,m+2} - b_{m-1,m+1} = 3m+6$$

for all $m > 0$. Taking $k = m + 2$ in (1) we get

$$(4) \quad (m + 3)a_{m,m+2} + b_{m,m+2} = (3m^2 + 15m + 30)/2$$

Remark 1. We have $b_{m,m+1} = 0$ and $a_{m,m+1} = m + 2$ for all m . From (4) we get that if m is even, then $a_{m,m+2} = 3m/2 + 3$ and $b_{m,m+2} = 1$, while if m is odd, then $a_{m,m+2} = 3m/2 + 5/2$ and $b_{m,m+2} = m/2 + 5/2$. Hence for all $m \geq 3$ we have $a_{m-1,m+1} > m$, $a_{m,m+2} = a_{m-1,m+1} + 2$ if m is even and $a_{m,m+2} = a_{m-1,m+1} + 1$ if m is odd. We have $\binom{m+6}{3} - \binom{m+2}{3} = 2m^2 + 12m + 10$ and hence $a_{m,m+3} = 2m + 4$ and $b_{m,m+3} = 4$ for all $m \geq 1$, $a_{0,3} = 6$, $b_{0,3} = 2$. We have $\binom{m+7}{3} - \binom{m+2}{3} = (5m^2 + 35m + 70)/2$. If m is even and $m \geq 6$, then $a_{m,m+4} = 5m/2 + 5$ and $b_{m,m+4} = 10$. If $m \in \{2, 4\}$, then $a_{m,m+4} = 5m/2 + 6$ and $b_{m,m+4} = 5 - m$. If m is odd and $m \geq 17$, then $a_{m,m+4} = 5m/2 + 9/2$ and $b_{m,m+4} = (m + 25)/2$. If $m \in \{3, 5, 7, 9, 11, 13, 15\}$, then $a_{m,m+4} = 5m/2 + 11/2$ and $b_{m,m+4} = (15 - m)/2$.

For all positive integers m, d the *critical value* of the pair (m, d) is the minimal integer $k \geq m$ such that $\binom{m+2}{3} + (k + 1)d \leq \binom{k+3}{3}$. Let $W \subset \mathbb{P}^3$ be a union of d disjoint lines with $P \notin W$. The scheme $mP \cup W$ has maximal rank if and only if $h^0(\mathcal{I}_{mP \cup W}(k - 1)) = 0$ and $h^1(\mathcal{I}_{mP \cup W}(k)) = 0$, where k is the critical value of (m, d) . Using (2) it is easy to check that for a fixed integer $m > 0$ the sequence $a_{m,k}$ is strictly increasing for all $k \geq m - 1$ (we have $a_{m,m-1} = 0$). The integer k is the critical value of the pair (m, d) if and only if $a_{m,k-1} < d \leq a_{m,k}$.

3. ASSERTIONS $B(m)$, $R(m)$ AND $H_{m,k}$

For every odd positive integer m we define Assertion $B(m)$ in the following way.

Assertion $B(m)$, $m \geq 1$, m odd: There is a 7-ple (Y, L, R, S, O, H, v) with the following properties:

- (1) H is a plane containing P , L and R are lines of H , $L \neq R$, $P \notin L \cup R$, and $\{O\} := L \cap R$;
- (2) Y is a union of $a_{m,m+2}$ disjoint lines, $P \notin Y$ and $Y \cap H$ is finite;
- (3) $S \subset H \cap Y$, $\sharp(S) = b_{m,m+2} - 2$;
- (4) v is a disjoint union of $b_{m,m+2} - 2$ tangent vectors of \mathbb{P}^3 , each of them with a point of S as its support;
- (5) $\sharp(S \cap L) = \lceil (m + 3)/4 \rceil$, $\sharp(S \cap R) = \lfloor (m + 3)/4 \rfloor$ and $L \cap R \cap S = \emptyset$;
- (6) $h^1(\mathcal{I}_{mP \cup Y \cup v \cup \{O\}}(m + 2)) = 0$.

Take (Y, L, R, S, O, H, v) satisfying the second, third and fourth of the conditions of $B(m)$. We have $h^0(\mathcal{O}_{mP \cup Y \cup v \cup \{O\}}(m + 2)) = \binom{m+5}{3} - 1$ and hence $h^1(\mathcal{I}_{mP \cup Y \cup v \cup \{O\}}(m + 2)) = h^0(\mathcal{I}_{mP \cup Y \cup v \cup \{O\}}(m + 2)) - 1$.

For every even integer $m \geq 2$ we define Assertion $B(m)$ in the following way.

Assertion $B(m)$, $m \geq 2$, m even: There is a quadruple (Y, L, R, H) with the following properties:

- (1) H is a plane containing P , L and R are lines of H , $L \neq R$, and $P \notin L \cup R$;
- (2) Y is a union of $a_{m,m+2}$ disjoint lines, $P \notin Y$ and $Y \cap H$ is finite;
- (3) $\sharp((Y \cap H) \cap L) = \lceil (m + 2)/4 \rceil$, $\sharp((Y \cap H) \cap R) = \lfloor (m + 2)/4 \rfloor$;
- (4) $h^1(\mathcal{I}_{mP \cup Y}(m + 2)) = 0$.

The last condition of $B(m)$, m even, is equivalent to $h^0(\mathcal{I}_{mP \cup Y}(m + 2)) = 1$.

Lemma 1. $B(m)$ is true for all $m \geq 2$.

Proof. We first prove $B(2)$. Let $Y \subset \mathbb{P}^3$ be a general union of 6 lines (hence $P \notin Y$). By [4, part (i)(e) of Theorem 4.2] we have $h^1(\mathcal{I}_{2P \cup Y}(4)) = 0$. Let $H \subset \mathbb{P}^3$ be a general plane though P . Moving Y we see that we may assume that no 3 of the points of $(Y \cap H) \cup \{P\}$ are collinear.

Now assume $m \geq 3$ and that $B(m-1)$ is true.

(a) In this step we assume that m is odd. Take (Y, L, R, S, H) satisfying $B(m-1)$. We have $h^1(\mathcal{I}_{(m-1)P \cup Y}(m+1)) = 1$. Let $D \subset H$ be a general line. Let $v \subset H$ be a union of tangent vectors of H with S as its support, but no tangent vector being a tangent vector of $L \cup R$. We first check that $h^1(\mathcal{I}_{mP \cup Y \cup D \cup v \cup \{O\}}(m+2)) = 0$. Since $\text{Res}_H(mP \cup Y \cup D \cup v \cup \{O\}) = (m-1)P \cup Y$ and $h^1(\mathcal{I}_{(m-1)P \cup Y}(m+1)) = 0$, it is sufficient to prove that $h^1(H, \mathcal{I}_{((mP \cup Y) \cap H) \cup D \cup v \cup \{O\}}(m+2)) = 0$, i.e. $h^1(H, \mathcal{I}_{(mP \cup Y) \cap H} \cup v \cup \{O\})(m+1)) = 0$. The scheme $((mP \cup Y) \cap H) \cup v \cup \{O\}$ is a general union of $\{mP, H\}$, the scheme $v \cup \{O\}$ and $a_{m-1, m+1} - (m+1)/2$ general points of H . Hence it has degree $\binom{m+1}{2} + 2(m+1)/2 + 3(m-1)/2 + 3 - (m+1)/2 + 1 = \binom{m+3}{2}$. We deform D in a flat family of lines outside H (we may do it even fixing either the point of $D \cap L$ or the point of $D \cap R$). For general v it is easy to check that $h^1(H, \mathcal{I}_{\{mP, H\} \cup v \cup \{O\}}(m+1)) = 0$ (order the points of S and then add the corresponding connected component v_i of v following the ordering first with the point P_i of S general in a component of $L \cup R$ and then with v_i general among the tangent vectors of H with P_i as its support; at each point use that $h^0(H, \mathcal{O}_{\{mP, H\}}(m)) = m+1$ and that if $P_i \in L_i$, then $|\mathcal{I}_{\{mP, H\} \cup \{2P_i, L_i\}}(m+1)| \cong |\mathcal{I}_{\{mP, H\}}(m)|$). Since $Y \cap H \setminus S$ is general in H , we get $h^i(H, \mathcal{I}_{((mP \cup Y) \cap H) \cup v \cup \{O\}}(m+1)) = 0$, $i = 0, 1$.

(b) In this step we assume that m is even. Take (Y, L, R, S, O, H, v) satisfying $B(m-1)$. Let $w \subset \mathbb{P}^3$ be a general tangent vector with O as its support. The scheme $Y \cup L \cup R \cup w \cup v$ is a flat limit of a family of disjoint unions of $a_{m, m+2}$ lines (i.e. there are a flat family $\{Y_t\}_{t \in \Gamma}$, Γ an integral affine curve, $o \in \Gamma$, $Y_o = Y \cup L \cup R \cup w \cup v$) such that $Y \subset Y_t$, say $Y_t = Y \cup L_t \cup R_t$ for all t with $\{L_t\}$, and $\{R_t\}$ flat families with $L_o = L$ and $L_o = L$, and either $L_t \cap L \neq \emptyset$ for all t , $R_t \cap R = \emptyset$ for all $t \neq o$ (case $m \equiv 2 \pmod{4}$) or $R_t \cap R \neq \emptyset$ for all t and $L_t \cap L = \emptyset$ for all $t \neq o$ (case $m \equiv 0 \pmod{4}$). We may take as the new set S the set $S \cup (L_t \cup R_t) \cap (L \cup R)$ for a general $t \in \Gamma$. By the semicontinuity theorem for cohomology ([5, III.12.8]) it is sufficient to prove that $h^1(\mathcal{I}_{mP \cup Y \cup L \cup R \cup v \cup w}(m+2)) = 0$. Since $(mP \cup Y \cup L \cup R \cup v \cup w) \cap H = \{mP, H\} \cup L \cup R \cup ((Y \cap H) \setminus S)$ and $\text{Res}_H(mP \cup Y \cup L \cup R \cup v \cup w) = (m-1)P \cup Y \cup v \cup \{O\}$, it is sufficient to prove that $h^1(H, \mathcal{I}_{\{mP, H\} \cup L \cup R \cup ((Y \cap H) \setminus S)}(m+2)) = 0$, i.e. $h^1(H, \mathcal{I}_{\{mP, H\} \cup ((Y \cap H) \setminus S)}(m)) = 0$. This is true, because $(Y \cap H) \setminus S$ is general in H and $\#((Y \cap H) \setminus S) = 3m/2 + 1 - m/2 = m+1 = h^0(H, \mathcal{I}_{\{mP, H\}}(m))$. \square

Remark 2. Fix (m, d) with critical value $m+1$ and degree $d \geq m+1$. Let $W \subset \mathbb{P}^m$ be a general union of d lines. Since $a_{m, m+1} = m+2$, we have $m+1 \leq d \leq m+2$. By [4, part (i)(c) of Theorem 4.2] the scheme $mP \cup W$ has maximal rank.

Lemma 2. Fix an integer $d \leq a_{m, m+2}$ and let $X \subset \mathbb{P}^3$ be a general union of d lines. Then $h^1(\mathcal{I}_{mP \cup X}(m+2)) = 0$.

Proof. This statement is obvious if $m = 1$ by [6]. Assume $m \geq 2$. It is sufficient to find a disjoint union W of d lines such that $P \notin W$ and $h^1(\mathcal{I}_{mP \cup W}(m+2)) = 0$. Take a solution of $B(m)$ and call Y the curve in it. Take as W the union of d of the lines of Y . \square

For all odd integer $m \geq 3$ let $R(m)$ denote the following assertion:

Assertion $R(m)$, m odd, $r \geq 3$: There exists a quintuple (Y, S, D, H, v) with the following properties:

- (1) $Y \subset \mathbb{P}^3$ is a disjoint union of $3m/2 + 5/2$ lines, $P \notin Y$, H is a plane containing P , $D \subset H$ is a smooth conic such that $P \notin D$ and $S := (Y \cap H) \cap D$ has cardinality $m/2 + 5/2$;
- (2) $v \subset \mathbb{P}^3$ is a disjoint union of tangent vectors of \mathbb{P}^3 with $v_{\text{red}} = S$; no connected component of v is contained in Y ;
- (3) $h^i(\mathcal{I}_{mP \cup Y \cup v}(m+2)) = 0$.

Lemma 3. $R(m)$ is true for all odd integers $m \geq 3$.

Proof. Take (Y, L, R, H) satisfying $B(m-1)$. We have $\sharp(Y \cap (L \cup R)) = (m+1)/2$, $h^1(\mathcal{I}_{(m-1)P \cup Y}(m+1)) = 0$ and $h^0(\mathcal{I}_{(m-1)P \cup Y}(m+1)) = 1$. Since $h^0(\mathcal{I}_{(m-2)P \cup Y}(m)) = 0$, $P \in H$, and $Y \cap (R \cup L) \neq Y \cap H$, there is $o \in L \cup R$ not in the base locus of $|\mathcal{I}_{(m-1)P \cup Y}(m+1)|$ and hence $h^i(\mathcal{I}_{(m-1)P \cup Y \cup \{o\}}(m+1)) = 0$, $i = 0, 1$. We may deform $(Y, L \cup R, o)$ to (Y', C, o') , where $C \subset H$ is a smooth conic, $P \notin C$, $o' \in C \setminus C \cap Y$, $\sharp(Y' \cap C) = (m+1)/2$ and $h^0(\mathcal{I}_{(m-1)P \cup Y \cup \{o'\}}(m+1)) = 0$. We may take as o' a general point of C . Let o'' be another general point of C and call T the line spanned by o' and o'' (alternatively, take a general line $T \subset H$ and set $\{o', o''\} := C \cap T$). Let $w \subset H$ be a general union of tangent vectors of H , each of them supported by a different point of $Y \cap (L \cup R)$. Let $v' \subset \mathbb{P}^3$ be a general tangent vector of \mathbb{P}^3 with o' as its support (hence $\text{Res}_H(v') = \{o'\}$). Let $v'' \subset H$ be a general tangent vector of H with $v''_{\text{red}} = \{o''\}$. Since $v'' \subset H$, we have $\text{Res}_H(v'') = \emptyset$. Since v'' is general, it is not tangent to T and hence $\text{Res}_T(v'') = \{o''\}$. Take $Y' := Y \cup T$, $v := w \cup v' \cup v''$ and $S := Y \cap (L \cup R) \cup \{o', o''\}$. We want to check that the quintuple (Y', S, C, H, v) satisfies $R(m)$. The scheme v is a union of tangent vectors, one for each point of S . We have $\sharp(S) = (m+1)/2 + 2 = (m+5)/2$. The set S is contained in the smooth conic D . It is sufficient to check that $h^i(\mathcal{I}_{mP \cup Y' \cup v}(m+2)) = 0$, $i = 0, 1$. Since $\text{Res}_H(mP \cup Y' \cup v) = (m-1)P \cup Y \cup \{o'\}$, $h^i(\mathcal{I}_{(m-1)P \cup Y \cup \{o'\}}(m+1)) = 0$, $i = 0, 1$, $o' \in D$, $(mP \cup Y' \cup v) \cap H = \{(m-1)P \cup w \cup T \cup \{o'\} \cup v'' \cup ((Y \cap H) \setminus (Y \cap H) \cap (L \cup R))\}$ and $\text{Res}_T(\{mP, H\} \cup T \cup v'' \cup (Y \cap H) \cup w) = \{mP, H\} \cup (Y \cap H) \cap w \cup \{o''\}$, it is sufficient to prove that $h^i(H, \mathcal{I}_{\{mP, H\} \cup (Y \cap H) \cup w \cup \{o''\}}(m+1)) = 0$. We have $\deg(\{mP, H\} \cup (Y \cap H) \cup w \cup \{o''\}) = \binom{m+1}{2} + 3m/2 + 3/2 + (m+1)/2 + 1 = \binom{m+3}{2}$. Use again that $h^1(H, \mathcal{I}_{\{mP, H\} \cup w}(m+1)) = 0$ (as in part (a) of the proof of Lemma 1) and that $Y \cap H \setminus v_{\text{red}}$ is general in H . \square

Lemma 4. Fix an integer $d > a_{m,m+2}$ and let $X \subset \mathbb{P}^3$ be a general union of d lines. Then $h^0(\mathcal{I}_{mP \cup X}(m+2)) = 0$.

Proof. It is sufficient to prove the lemma when $d = a_{m,m+2} + 1$. First assume that m is even. Take a solution (Y, L, R, H) of $B(m)$. Since $h^0(\mathcal{I}_{mP \cup Y}(m+2)) = 1$, we have $h^0(\mathcal{I}_{mP \cup Y \cup D}(m+2)) = 0$ for any line D through a general point of \mathbb{P}^3 .

Now assume that m is odd. Let $W \subset \mathbb{P}^3$ be a general union of $a_{m-1,m+1}$ lines. Since $B(m-1)$ is true, we have $h^1(\mathcal{I}_{(m-1)P \cup W}(m+1)) = 0$ and hence $h^0(\mathcal{I}_{(m-1)P \cup W \cup o}(m+1)) = 0$ for a general $o \in \mathbb{P}^3$. Let $M \subset \mathbb{P}^3$ be a general plane containing $\{P, o\}$. Let $L', R' \subset M$ be two general lines through o . It is sufficient to prove that $h^0(\mathcal{I}_{mP \cup W \cup L' \cup R' \cup 2o}(m+2)) = 0$. Since $\text{Res}_M(mP \cup W \cup L' \cup R' \cup 2o) = (m-1)P \cup W \cup \{o\}$, it is sufficient to prove that $h^0(H, \mathcal{I}_{\{mP, H\} \cup (W \cap H) \cup L' \cup R'}(m +$

2)) = 0, i.e. $h^0(H, \mathcal{I}_{\{mP, H\} \cup (W \cap H)}(m)) = 0$. Since $W \cap H$ is a general union of $a_{m-1, m+1} > m$ points of H , we have $h^0(H, \mathcal{I}_{\{mP, H\} \cup (W \cap H)}(m)) = 0$. \square

Consider the following statement:

Assertion $H_{m,k}$, $m > 0$, $k \geq m + 2$: There exist a quintuple (Y, Q, S, v, E) with the following properties:

- (1) $Y \in L(P, a_{m,k})$, Q is a smooth quadric surface intersecting transversally Y , $P \notin Q$;
- (2) $S \subseteq Y \cap Q$, $\#(S) = b_{m,k}$ and $v \subset \mathbb{P}^3$ is a disjoint union of tangent vectors with $v_{\text{red}} = S$ and no connected component of v contained in Y ;
- (3) $E \subset Q$ is a disjoint union of $\lceil b_{m,k}/2 \rceil$ lines, $S \subset E$ and each component of E contains at most two points;
- (4) $h^i(\mathcal{I}_{mP \cup Y \cup v}(k)) = 0$, $i = 0, 1$.

Take (Y, Q, S, v, E) satisfying the first two conditions of the definition of $H_{m,k}$. We have $h^0(\mathcal{O}_{mP \cup Y \cup v}(k)) = \binom{k+3}{3}$ and hence $h^0(\mathcal{I}_{mP \cup Y \cup v}(k)) = h^1(\mathcal{I}_{mP \cup Y \cup v}(k))$. Now assume that (Y, Q, S, v, E) satisfies the third condition of the definition of $H_{m,k}$. If $b_{m,k}$ is even, then each line of S contains exactly two points of S . If $b_{m,k}$ is odd, then $\#(S \cap L) = 2$ for $(b_{m,k} - 1)/2$ of the components of E , while $\#(S \cap L) = 1$ for the other component.

From now on $Q \subset \mathbb{P}^3$ is a smooth quadric surface such that $P \notin Q$.

Lemma 5. $H_{m, m+3}$ is true for all $m > 0$.

Proof. We have $a_{m, m+1} = m + 2$, $b_{m, m+1} = 0$, $a_{m, m+3} = 2m + 4$ and $b_{m, m+3} = 4$ (Remark 1). Let $Y \subset \mathbb{P}^3$ be a general union of $m + 2$ lines. By [4, Part (i)(c) of Theorem 4.2] we have $h^i(\mathcal{I}_Y(m+1)) = 0$, $i = 0, 1$. For a general Y we may assume that $Y \cap Q$ is formed by $2m + 4$ general points of Q . Let $F \subset Q$ be a general union of $m + 2$ lines of type $(0, 1)$. Fix $S_1 \subset Y \cap Q$ such that $\#(S_1) = 2$. Let $E' \subset Q$ be the union of the lines of type $(1, 0)$ containing a point of S_1 . Fix $S_2 \subset E' \cap F$ such that $\#(S_2 \cap L) = 1$ for each component L of E' and that no component of F contains two points of S_2 . Set $S := S_1 \cup S_2$ and call $v \subset Q$ a general union of tangent vectors of Q with S as its support. We claim that $h^i(\mathcal{I}_{mP \cup Y \cup F \cup v}(m+3)) = 0$. Since $\text{Res}_Q(mP \cup Y \cup F \cup v) = mP \cup Y$, $Q \cap (mP \cup Y \cup F \cup v) = F \cup v \cup ((Y \cap Q) \setminus S_1)$, $\text{Res}_F(F \cup v \cup ((Y \cap Q) \setminus S_1)) = ((Y \cap Q) \setminus S_1) \cup v_{\text{red}} = Y \cap Q \cup S_2$ and $h^i(\mathcal{I}_{mP \cup Y}(m+1)) = 0$, $i = 0, 1$, to prove the claim it is sufficient to prove that $h^i(Q, \mathcal{I}_{((Y \cap Q) \setminus S_1) \cup v}(m+3, 1)) = 0$, $i = 0, 1$. We have $\deg(((Y \cap Q) \setminus S_1) \cup S) = 2m + 6$. Hence it is sufficient to use that $h^1(Q, \mathcal{I}_S(m+3, 1)) = 0$ (S is the union of two degree 2 schemes on two different lines of type $(1, 0)$ and $(Y \cap Q) \setminus S_1$ is general in Q). \square

Lemma 6. $H_{m, m+4}$ is true for all $m \geq 2$.

Proof. The proof depends on the parity of m .

(a) First assume that m is even. We have $a_{m, m+2} = 3m/2 + 3$ and $b_{m, m+2} = 1$.

(a1) Assume for the moment $m \geq 6$ and hence $a_{m, m+4} = 5m/2 + 5$ and $b_{m, m+4} = 10$ (Remark 1). Let $Y \subset \mathbb{P}^3$ be a general union of $a_{m, m+2} = 3m/2 + 3$ lines. Since $B(m)$ is true (Lemma 1) and $b_{m, m+2} = 1$, we have $h^1(\mathcal{I}_{mP \cup Y}(m+2)) = 0$ and $h^0(\mathcal{I}_{mP \cup Y}(m+2)) = 1$. The last equality implies $h^0(\mathcal{I}_{mP \cup Y}(m+1)) = 0$.

Let $T \subset \mathbb{P}^3$ be the only surface of degree $m+2$ containing $mP \cup Y$. Fix a system x_0, x_1, x_2, x_3 of homogeneous coordinates and let $f(x_0, x_1, x_2, x_3)$ be a degree $m+2$ homogeneous equation of T . In characteristic zero we have $\partial f / \partial x_i \neq 0$ for at least

one index i . Since $\partial f / \partial x_i \neq 0$ and $h^0(\mathcal{I}_{mP \cup Y}(m+1)) = 0$, we have $\partial f / \partial x_i|_{mP \cup Y} \neq 0$, i.e. $mP \cup Y \not\subseteq \text{Sing}(T)$, i.e. $Y \not\subseteq \text{Sing}(T)$. In characteristic $p > 0$ we need to prove the existence of $Y \in L(P, 3m/2 + 3)$ such that $h^0(\mathcal{I}_{mP \cup Y}(m+2)) = 1$ and at least one component of Y is not contained in the singular locus of the only degree m hypersurface containing $mP \cup Y$. We get this using the proof that $B(m-1)$ implies $B(m)$ when m is even (part (b) of the proof of Lemma 1). Take (Y, L, R, S, O, H, v) satisfying $B(m-1)$ and use that $h^0(H, \mathcal{I}_{\{mP, H\} \cup 2L \cup 2R}(m+2)) = 0$.

Since $Y \not\subseteq \text{Sing}(T)$, there is $O \in Y$ and a tangent vector w to \mathbb{P}^3 with $w \not\subseteq T$. Hence $h^0(\mathcal{I}_{mP \cup Y \cup w}(m+2)) = 0$ and so $h^1(\mathcal{I}_{mP \cup Y \cup w}(m+2)) = 0$. Take as Q a general quadric surface through O . We have $P \notin Q$ and $\text{Res}_Q(mP \cup Y \cup w) = mP \cup Y \cup w$. Moving the lines of Y among the unions of $a_{m,m+2}$ disjoint lines of \mathbb{P}^3 , one of them containing O , we may assume that $(Y \cap Q) \setminus \{O\}$ is a general union of $3m+5$ points of Q . Let $F \subset Q$ be a union of $m+2$ distinct lines of type $(0,1)$ of Q with $\{O\} = Y \cap F$. Fix $S_1 \subset (Y \cap Q) \setminus \{O\}$ with $\#(S_1) = 5$. Let $E \subset Q$ be the union of the 5 lines of type $(1,0)$ of Q containing one point of S_1 . Take $S_2 \subset E \cap F$ such that each line of E contains exactly one point of S_2 and each line of F contains at most one point of S_2 . Set $S := S_1 \cup S_2$. Let $v \subset Q$ be a general union of tangent vectors of Q with $v_{\text{red}} = S$. As in the proof of Lemma 5 it is sufficient to prove that $h^i(\mathcal{I}_{mP \cup Y \cup w \cup F \cup v}(m+4)) = 0$, $i = 0, 1$. Since $h^i(\mathcal{I}_{mP \cup Y \cup w}(m+2)) = 0$, $i = 0, 1$, it is sufficient to prove that $h^i(Q, \mathcal{I}_{(Y \cap Q) \cup F \cup v}(m+4)) = 0$, i.e. $h^i(Q, \mathcal{I}_{(Y \cap Q) \cup v}(m+4, 2)) = 0$, $i = 0, 1$. We have $\deg((Y \cap Q) \cup v) = 20 + (3m+6) - 10$. Since v is general, $\deg(v) = 20$ and S is general with the only restriction that 5 lines of type $(1,0)$ of Q contain each two points of S , we have $h^1(\mathcal{I}_v(m+4, 2)) = 0$. Since $Y \cap Q \setminus S_1$ is general in Q , we get $h^i(Q, \mathcal{I}_{(Y \cap Q) \cup v}(m+4, 2)) = 0$, $i = 0, 1$.

(a2) Now assume $m \in \{2, 4\}$. We have $a_{m,m+4} = 5m/2 + 6$ and $b_{m,m+4} = 5 - m$. Now F is a union of $m+3$ lines, $\#(S_1) = 2 - m/2$, $\deg(E) = 3 - m/2$ and $\#(S_2) = 3 - m/2$.

(b) Now assume that m is odd.

(b1) Assume $m \geq 17$. We have $a_{m,m+2} = 3m/2 + 5/2$ and $b_{m,m+2} = (m+5)/2$. Since $m \geq 17$, we have $a_{m,m+4} = 5m/2 + 9/2$ and $b_{m,m+4} = (m+25)/2$ (Remark 1). Take a solution (Y, S, D, H, v) of $R(m)$ (Lemma 3). Let $Q \subset \mathbb{P}^3$ be a general quadric surface containing D . Since $P \notin D$, then $P \notin Q$. The quadric Q is smooth and it intersects transversally Q . By the semicontinuity theorem for cohomology ([5, III.12.8]) for general v we may also assume that no connected component of v is contained in Q , i.e. that $(mP \cup Y \cup v) = Y \cap Q$ (as schemes) and that $\text{Res}_Q(mP \cup Y \cup v) = mP \cup Y \cup v$. We may move each component of Y keeping fixed its point in D . Hence we may assume that $Y \cap (Q \setminus D)$ is a general subset of Q with cardinality $2a_{m,m+2} - b_{m,m+2}$. We have $a_{m,m+4} - a_{m,m+2} = m+2 \geq (m+5)/2 = b_{m,m+2}$. Let $F \subset Q$ be a disjoint union of $m+2$ lines of type $(0,1)$ with the only restriction that $F \cap Y = Y \cap D$ (it exists, because $m+2 \geq b_{m,m+2}$). Fix $S_1 \subseteq (Y \cap Q) \setminus S$ such that $\#(S_1) = \lfloor b_{m,m+4}/2 \rfloor$ (it exists because $2a_{m,m+2} = 3m+5 \geq (m+5)/2 + m \geq b_{m,m+2} + \lfloor b_{m,m+4}/2 \rfloor$). Let E_1 be the union of the lines of type $(1,0)$ of Q containing one point of S_1 . If $b_{m,m+4}$ is even, then set $E' := E$. If $b_{m,m+4}$ is odd, then let E' be the union of E_1 and a general line of type $(1,0)$ of Q . Let $S_2 \subset E' \cap E''$ be the union of one point for each component of E' , with the restriction that $S_2 \cap S_1 = \emptyset$ and that each point of S_2 is contained in a different line of E'' ; we may find such a set S_2 , because $E'' \cap S_1 = \emptyset$ and

$\deg(E'') = a_{m,m+4} - a_{m,m+2} = m+2 \geq \lceil b_{m,m+4}/2 \rceil$. Let $v' \subset Q$ be a general union of $b_{m,m+4}$ tangent vectors of Q with $v'_{\text{red}} = S'$. Since v' is general, no connected component of v' is contained in E'' (hence $\text{Res}_{E''}((Y \cap Q) \cup v') = Y \cap (Q \setminus E'') \sqcup S' = ((Y \cap Q) \setminus S) \sqcup S'$).

(b2) Assume $m \in \{3, 5, 7, 9, 11, 13, 15\}$. We have $a_{m,m+4} - a_{m,m+2} = m+3$ and $b_{m,m+4} = (15-m)/2$. We make the construction of step (b1) with $\deg(F) = m+3$, $\sharp(S_1) = \lfloor (15-m)/4 \rfloor$, and $\deg(E) = \sharp(S_2) = \lceil (15-m)/4 \rceil$. \square

Lemma 7. *For all integers $k \geq m+3$ we have $a_{m,k} - a_{m,k-2} \geq a_{0,k} - a_{0,k-2} - 1 \geq \lceil k/2 \rceil$.*

Proof. From (2) and the same equation for $m=0$ we get

$$\begin{aligned} 2a_{m,k-2} + (k+1)(a_{m,k} - a_{m,k-2}) + b_{m,k} - b_{m,k-2} = \\ 2a_{0,k} + (k+1)(a_{0,k} - a_{0,k-2}) + b_{0,k} - b_{0,k-2} \end{aligned}$$

We have $b_{0,x} = 0$ if $x \equiv 0, 1 \pmod{3}$ and $b_{0,x} = (x+1)/3$ if $x \equiv 2 \pmod{3}$. The definitions of the integers $a_{m,k-2}$ and $a_{0,k-2}$ give $a_{m,k-2} \leq a_{0,k-2}$, proving the lemma. \square

Lemma 8. *$H_{m,k}$ is true for all $k \geq m+3$.*

Proof. By Lemmas 5 and 6 we may assume $k \geq m+5$ and that $H_{m,k-2}$ is true. Fix a solution (Y, Q, S, v, E) of $H_{m,k-2}$. Deforming if necessary each line of Y we may assume that $(Q \cap Y) \setminus S$ is a general subset of Q . Taking instead of v a union of general tangent vectors of \mathbb{P}^3 with the points of S as their support we may assume that no connected component of v is contained in Q . Therefore $\text{Res}_Q(Y \cup v) = Y \cup v$ and $(Y \cup v) \cap Q = Y \cap Q$ (as schemes). Call $(0, 1)$ the ruling of Q containing E (any ruling of Q if $b_{m,k-2} = 0$ and hence $E = \emptyset$). Lemma 7 gives $a_{m,k} - a_{m,k-2} \geq \deg(E)$. Let $F \subset Q$ be a general union of $a_{m,k} - a_{m,k-2} - \lceil b_{m,k-2}/2 \rceil$ lines of type $(0, 1)$ of Q . Set $E'' := E \cup F$.

Claim 1: If $k \geq m+5$, then $2a_{m,k-2} \geq k-2 + k/2$.

Proof of Claim 1: We have $2(k-1)a_{m,k-2} + 2b_{m,k-2} = 2\binom{k+3}{3} - 2\binom{m+2}{3}$ and $b_{m,k-2} \leq k-2$. Set $\psi(k, m) := 2\binom{k+3}{3} - 2\binom{m+2}{3} - (k-1)(k-2 + k/2) - 2k + 4$. It is sufficient to prove that $\psi(k, m) \geq 0$ for all $k \geq m+5$. We have $\psi(m+5, m) = (m+8)(m+7)(m+6)/3 - (m+2)(m+1)m/3 - (m+4)(3m+11)/2 - 2m + 6 \geq 0$ and $\psi(k+1, m) \geq \psi(k, m)$ for all $k \geq m+5$.

Fix $S_1 \subseteq (Y \cap Q) \setminus S$ such that $\sharp(S_1) = \lfloor b_{m,k} \rfloor$ (it exists by Claim 1 and the inequalities $b_{m,k-2} \leq k-2$, $b_{m,k} \leq k$). Let $E_1 \subset Q$ be the union of the lines of type $(1, 0)$ of Q containing one point of S_1 . If $b_{m,k}$ is even, then set $E' := E$. If $b_{m,k}$ is odd, then let E' be the union of E_1 and a general line of type $(1, 0)$ of Q . Let $S_2 \subset E' \cap E''$ be the union of one point for each component of E' , with the restriction that $S_2 \cap S_1 = \emptyset$ and that each point of S_2 is contained in a different line of E'' ; we may find such a set S_2 , because $E'' \cap S_1 = \emptyset$ and $\deg(E'') = a_{m,k} - a_{m,k-2} \geq \lceil b_{m,k}/2 \rceil$ (Lemma 7). Set $S' := S_1 \cup S_2$. Let $v' \subset Q$ be a general union of $b_{m,k}$ tangent vectors of Q with $v'_{\text{red}} = S'$. Since v' is general, no connected component of v' is contained in E'' (hence $\text{Res}_{E''}((Y \cap Q) \cup v') = Y \cap (Q \setminus E'') \sqcup S' = ((Y \cap Q) \setminus S) \sqcup S'$).

Claim 2: We claim that $h^i(\mathcal{I}_{mPY \cup v \cup E'' \cup v'}(k)) = 0$, $i = 0, 1$.

Proof of Claim 2: Since $\text{Res}_Q(mPY \cup v \cup E'' \cup v') = mPY \cup v$, $Q \cap (mPY \cup v \cup E' \cup v') = (Y \cap Q) \cup E'' \cup v'$ and $h^i(\mathcal{I}_{mPY \cup v}(k-2)) = 0$, $i = 0, 1$, it is sufficient to prove that $h^i(Q, \mathcal{I}_{(Y \cup Q) \cup v' \cup E'}(k)) = 0$, $i = 0, 1$, i.e. $h^i(Q, \mathcal{I}_{((Y \cap Q) \setminus S) \cup S'}(k, k -$

$a_{m,k} + a_{m,k-2}) = 0$, $i = 0, 1$. By (2) we have $\sharp((Y \cap Q) \setminus S) \cup S' = (k+1)(k+1 - a_{m,k} + a_{m,k-2})$. Hence it is sufficient to prove that the set $((Y \cap Q) \setminus S) \cup S'$ gives independent conditions to the linear system $|\mathcal{O}_Q(k, k - a_{m,k} + a_{m,k-2})|$. Since $(Y \cap Q) \setminus S$ is general in Q , it is sufficient to prove that S' gives independent conditions to $|\mathcal{O}_Q(k, k - a_{m,k} + a_{m,k-2})|$. This is true since S_1 is general and hence the only restriction on the subset S' of Q is that each line of E_1 contains two points of S' .

We may deform $Y \cup v \cup E''$ to a family of members of $L(P, a_{m,k})$ containing the points of S' and whose general member, Y' , intersects transversally Q , because each line of E'' contains at most one point of Q . The quintuple (Y', Q, S', v', E') satisfies $H_{m,k}$. \square

Proof of Theorem 1: Fix positive integers m, d with critical value k . Hence $a_{m,k-1} < d \leq a_{m,k}$. See Remark 2 and Lemma 2 for the cases $k = m, m+1, m+2$. Hence we may assume $k \geq m+3$ and that the theorem is true for the integers d such that (m, d) has critical value $< k$. Since $L(P, d)$ is irreducible, it is sufficient to prove the existence of $A, B \in L(P, d)$ such that $h^0(\mathcal{I}_{mP \cup B}(k-1)) = 0$ and $h^1(\mathcal{I}_{mP \cup A}(k)) = 0$. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface such that $P \notin Q$.

(a) In this step we prove the existence of A . Since any element of $L(P, d)$ is a union of some of the connected components of an element of $L(P, a_{m,k})$, it is sufficient to do the case $d = a_{m,k}$. Fix a solution (Y, Q, S, v, E) of $H_{m,k-2}$. Deforming if necessary each line of Y we may assume that $(Q \cap Y) \setminus S$ is a general subset of Q . Taking instead of v a union of general tangent vectors of \mathbb{P}^3 with the points of S as their support we may assume that no connected component of v is contained in Q . Therefore $\text{Res}_Q(Y \cup v) = Y \cup v$ and $(Y \cup v) \cap Q = Y \cap Q$ (as schemes). Call $(0, 1)$ the ruling of Q containing E (any ruling of Q if $b_{m,k-2} = 0$ and hence $E = \emptyset$). Lemma 7 gives $a_{m,k} - a_{m,k-2} \geq \deg(E)$. Let $F \subset Q$ be a general union of $a_{m,k} - a_{m,k-2} - \lceil b_{m,k-2}/2 \rceil$ lines of type $(0, 1)$ of Q . The scheme $Y \cup E \cup F \cup v$ is a flat limit of a family of disjoint unions of $a_{m,k}$ lines, none of them containing P . By the semicontinuity theorem for cohomology to prove the existence of A it is sufficient to prove that $h^1(\mathcal{I}_{mP \cup Y \cup E \cup F \cup v}(k)) = 0$. We proved a more difficult vanishing in the proof of Lemma 8 (copy it without v').

(b) In this step we prove the existence of B . By Lemma 4 we may assume $k-1 \geq m+3$. Since $b > a_{m,k-1}$, to prove the existence of B it is sufficient to prove it when $d = a_{m,k-1} + 1$.

(b1) Assume for the moment $k-1 \geq m+4$, i.e. $k-3 \geq m+2$. Fix a solution (Y, Q, S, v, E) of $H_{m,k-3}$. Deforming if necessary each line of Y we may assume that $(Q \cap Y) \setminus S$ is a general subset of Q . Taking instead of v a union of general tangent vectors of \mathbb{P}^3 with the points of S as their support we may assume that no connected component of v is contained in Q . Therefore $\text{Res}_Q(Y \cup v) = Y \cup v$ and $(Y \cup v) \cap Q = Y \cap Q$ (as schemes). Call $(0, 1)$ the ruling of Q containing E (any ruling of Q if $b_{m,k-3} = 0$ and hence $E = \emptyset$). Lemma 7 gives $a_{m,k-1} - a_{m,k-3} \geq \deg(E)$. Let $F \subset Q$ be a general union of $a_{m,k} - a_{m,k-3} - \lceil b_{m,k-2}/2 \rceil + 1$ lines of type $(0, 1)$ of Q . The scheme $Y \cup E \cup F \cup v$ is a flat limit of a family of disjoint unions of $a_{m,k-1} + 1$ lines, none of them containing P . By the semicontinuity theorem for cohomology to prove the existence of B it is sufficient to prove that $h^0(\mathcal{I}_{mP \cup Y \cup E \cup F \cup v}(k-1)) = 0$. Since $\text{Res}_Q(mP \cup Y \cup F \cup E \cup v) = mP \cup Y \cup E \cup v$ and $Q \cap (mP \cup Y \cup F \cup E \cup v) = (Y \cap Q) \cup E \cup F$, it is sufficient to prove that $h^0(Q, \mathcal{I}_{F \cup E \cup (Y \cap Q)}(k-1)) = 0$, i.e. that $h^0(Q, \mathcal{I}_{(Y \cap Q) \setminus S}(k-1, k - a_{m,k-1} + a_{m,k-3} - 2)) = 0$. Since $(Y \cap Q) \setminus S$ is general in

Q , it is sufficient to prove that $\sharp(Y \cap Q) - \sharp(S) \geq k(k - a_{m,k-1} + a_{m,k-3} - 1)$. By (2) for the integer $k' := k - 1$ we have $\sharp(Y \cap Q) - \sharp(S) = k(k - a_{m,k-1} + a_{m,k-3} - 1) + k - b_{m,k-1} > k(k - a_{m,k-1} + a_{m,k-3} - 1)$.

(b2) Now assume $k = m + 4$. We modify the proof of $H_{m,m+3}$ (Lemma 5). We have $a_{m,m+1} = m + 2$, $b_{m,m+1} = 0$, $a_{m,m+3} = 2m + 4$ and $b_{m,m+3} = 4$ (Remark 1). Let $Y \subset \mathbb{P}^3$ be a general union of $m + 2$ lines. By [4, Part (i)(c) of Theorem 4.2] we have $h^i(\mathcal{I}_Y(m + 1)) = 0$, $i = 0, 1$. For a general Y we may assume that $Y \cap Q$ is formed by $2m + 4$ general points of Q . Let $F \subset Q$ be a general union of $m + 3$ lines of type $(0, 1)$. Use $Y \cup F$. Since $Y \cap Q$ is a general subset of Q with cardinality $2m + 4$, we have $h^0(Q, \mathcal{I}_{Q \cap Y}(m + 3, 0)) = 0$. We also have $Y \cap F = \emptyset$ and hence $h^0(Q, \mathcal{I}_{(Y \cap Q) \cup F}(m + 3)) = 0$. \square

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